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On the instability of viscous flow in a rapidly rotating pipe

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The stability of almost fully developed viscous flow in a rotating pipe is considered. In cylindrical polar co-ordinates (r, ϕ, z) this flow has the velocity components

$$\{W_0 o(1),\quad \Omega r[1+o(\epsilon)],\quad W_0[1-r^2/r_0^2+o(1)]\},‡$$

where $\epsilon = W_0/2\Omega r_0$, and is bounded externally by the rigid cylinder $r = r_0$, which rotates about its axis with angular velocity Ω. In the limit of small ϵ, the disturbance equations can be solved in terms of Bessel functions, and it is shown that, in that limit, the flow is unstable for Reynolds numbers $R = W_0 r_0/\nu$ greater than $R_c \approx 82{\cdot}9$. The unstable disturbances take the form of growing spiral waves, which are stationary relative to the rotating cylinder, and the critical disturbance at $R = R_c$ has azimuthal wave-number 1 and axial wavelength $2\pi r_0/\epsilon$. Furthermore, it is shown that the most rapidly growing disturbance for $R > R_c$ has an azimuthal wave-number which increases with R. Some of the problems involved in testing the results by experiment are discussed, and a possible application to the theory of vortex breakdown is mentioned. In an appendix this instability is shown to be an example of inertial instability.

1. Introduction

It was shown in Pedley (1968), that a cylindrically symmetric shear flow of an incompressible fluid, such as Poiseuille flow in a circular pipe, is unstable to infinitesimal, inviscid disturbances when it is subjected to a rapid, almost rigid, rotation about its axis. The velocity components in cylindrical polar co-ordinates (r, ϕ, z) of the specific flow discussed there are

$$\{0,\ \Omega r g(r/r_0),\ W_0 f(r/r_0)\}, \tag{1.1}$$

where $f(x)$ is a function of order one and $g(x) \equiv 1+O(\epsilon^2)$.† It is bounded externally by the rigid cylinder $r = r_0$ and internally by either the axis $r = 0$ or another rigid cylinder $r = r_1 < r_0$. This flow is unstable for sufficiently small values of

$$\epsilon = W_0/2\Omega r_0, \tag{1.2}$$

if $f'(x)$ is non-zero for some x in the range $(r_1/r_0, 1)$. The disturbances to which the flow is unstable are stationary relative to a frame of reference rotating with

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‡ The symbols O, o have their usual meanings and refer to the limit $\epsilon \to 0$ except where explicitly described otherwise.

angular velocity Ω. They are also non-axisymmetric with azimuthal wave-number n, say, and with axial wave-number k such that $kr_0 = \epsilon\beta n$, where β is any number such that $\beta[\beta - f'(x)/x]$ is negative for some x in $(r_1/r_0, 1)$. The flow is clearly *stable* to axisymmetric disturbances, from the generalization of Rayleigh's criterion given by Howard & Gupta (1962) for swirling flow with axial shear. The non-axisymmetric instability is, nevertheless, of the same dynamical nature as the axisymmetric instability associated with violation of Rayleigh's criterion, often called inertial instability (see appendix). In the particular case of Poiseuille flow in a pipe, for which $f(x) \equiv 1 - x^2$ and the inner boundary is $x = 0$, it was shown in Pedley (1968) that the most rapidly growing disturbances are those for which $\beta = -1$ and $|n| \to \infty$.

In this paper, we examine the instability of such rotating Poiseuille flow (slightly generalized) in a *viscous* fluid. The undisturbed velocity components in the ϕ- and z-directions are given by (1.1) and by

$$f(x) \equiv 1 - x^2 + O(\epsilon^{a_1}), \quad g(x) \equiv 1 + O(\epsilon^{1+a_2}), \tag{1.3}$$

where $a_1 > 0$, $a_2 > 0$, and the pipe $r = r_0$ rotates about its axis with angular velocity Ω. In order that the postulated basic flow should itself be a steady solution of the Navier–Stokes equations, a radial velocity component U must in general be included, say $U = \epsilon^{a_1} W_0 h(x)$, where $h(x) = O(1)$ as $\epsilon \to 0$.

Experience with other problems in hydrodynamic stability suggests that the type of instability found in Pedley (1968) for an inviscid fluid will still be manifest in a viscous fluid if the Reynolds number,

$$R = W_0 r_0/\nu, \tag{1.4}$$

is sufficiently large, where ν is the kinematic viscosity of the fluid. Hence we expect there to be a critical Reynolds number, $R = R_c$, for which the flow is neutrally stable. For a given Reynolds number greater than R_c, however, it is unlikely that the most rapidly growing disturbance has an infinite azimuthal wave-number n; rather, we expect that it will occur for a specific, finite value of n. Both these expectations are confirmed below.

Clearly, the presence of viscosity may also introduce a totally different type of instability at large Reynolds numbers, associated perhaps with a 'critical layer' in which the fluid velocity is equal to the phase velocity of the disturbance. Here, however, we shall ignore this possibility, since such modes of instability are unlikely to occur at Reynolds numbers as low as the value ultimately obtained for R_c.

It happens that the solution of the problem can be obtained in closed form: the value of R_c, and the details of the most rapidly growing disturbance for $R > R_c$, are given by the roots of a certain transcendental equation, involving Bessel functions of complex argument. The analytical solution is presented in §§2–4 below, and the numerical results, guided by a further analytical theorem, are obtained in §5.

In §6 the results are evaluated in the light of some experiments, reported by other authors, with which they appear to conflict. The present theory is shown to be inapplicable to those experiments, and some of the problems to be overcome in a future experiment are discussed in detail. The section concludes with some remarks on the possible relevance of this type of hydrodynamic instability to the theory of vortex breakdown.

2. Reduction of the perturbation equations

If we assume infinitesimal perturbations to the basic flow given by (1.1) and (1.3), we can linearize the equations of motion, and may restrict our attention to a single Fourier component of the perturbation velocity $(\tilde{u}, \tilde{v}, \tilde{w})$ and pressure $\tilde{p}$. These quantities may therefore be written:

$$(\tilde{u}, \tilde{v}, \tilde{w}; \tilde{p}) = U_0[(1/x)y(x),\, v(x),\, w(x); 2\rho\Omega r_0 p(x)]\exp[i(\sigma t + n\phi + kz)], \quad (2.1)$$

where $x = r/r_0$, k is real, n is an integer (which with no loss of generality we may take to be non-negative), ρ is the uniform density of the fluid, and U_0 is a scaling factor which is arbitrary except in so far that it must be much smaller than W_0.

Let us define the following non-dimensional quantities:

$$\alpha = kr_0, \quad s(x) = \frac{n^2}{x^2} + \alpha^2, \quad \omega(x) = \frac{\sigma}{2\Omega} + \tfrac{1}{2}ng(x) + \epsilon\alpha f(x). \quad (2.2)$$

The radial, azimuthal, and axial equations of motion, and the equation of continuity, now become respectively

$$i\omega y - xgv + \epsilon^{1+a_1}[xh(y/x)' + yh'] = -xp' + \frac{\epsilon}{R}\left(y'' - \frac{y'}{x} - sy - \frac{2inv}{x}\right), \quad (2.3)$$

$$i\omega v + \frac{y}{2x^2}(x^2 g)' + \epsilon^{1+a_1}hv' = -\frac{inp}{x} + \frac{\epsilon}{R}\left(v'' + \frac{v'}{x} - \frac{v}{x^2} - sv + \frac{2iny}{x^3}\right), \quad (2.4)$$

$$i\omega w + \epsilon f'\frac{y}{x} + \epsilon^{1+a_1}hw' = -i\alpha p + \frac{\epsilon}{R}\left(w'' + \frac{w'}{x} - sw\right), \quad (2.5)$$

$$y' + inv + i\alpha xw = 0, \quad (2.6)$$

where a prime denotes differentiation with respect to x. The parameters ϵ and R are defined by (1.2) and (1.4) respectively; so far no assumption has been made about their orders of magnitude.

Boundary conditions must be applied to the perturbation quantities on the axis $r = 0$ and on the rigid boundary $r = r_0$. The no-slip condition on $r = r_0$ requires that all components of the perturbation velocity should be zero there; that is,

$$y(1) = v(1) = w(1) = 0. \quad (2.7)$$

The condition at $r = 0$ is that no physical quantity should be singular at that point, and is unaffected by the presence or absence of viscosity. This can be expressed by means of the following set of conditions, which are not necessarily independent (see Batchelor & Gill 1962):

$$\left.\begin{aligned} &w(0) = p(0) = 0 && \text{for} \quad n \neq 0,\\ &y(0) = y'(0) = v(0) = 0 && \text{for} \quad n \neq 1,\\ &y(0) = 0, \quad y'(0) = -inv(0) && \text{for} \quad n = 1. \end{aligned}\right\} \quad (2.8)$$

(Note that in Pedley (1968), §3, it was stated that $y = o(x)$ as $x \to 0$, which is wrong if $n = 1$. In that case it should be replaced by $y = o(1)$ as $x \to 0$; thus unstable disturbances with $n = 1$ are not prohibited.) The set of equations and boundary conditions (2.3)–(2.8) is of the sixth-order, and represents an eigenvalue problem for the quantity σ: if for certain values of n and α there is a solution for which σ has a negative imaginary part, then the flow is unstable to the disturbance defined by those values.

Now we shall make approximations based on the assumption that ϵ is small. First, we replace $f(x)$ and $g(x)$ by their functional forms (1.3). We are restricting our attention to the type of instability described in Pedley (1968), where only those disturbances for which n is non-zero and $|\alpha/n|$ is small are unstable; so next we set $\alpha = \epsilon\beta n$, where β is of order one. Thus

$$s(x) = \frac{n^2}{x^2} + O(\epsilon^2), \quad \omega(x) = \frac{\sigma}{2\Omega} + \frac{n}{2} + o(\epsilon) = \overline{\omega} + o(\epsilon), \tag{2.9}$$

say, where $\overline{\omega}$ is a constant, and the continuity equation (2.6) becomes

$$y' + inv + i\epsilon\beta nxw = 0. \tag{2.10}$$

If we eliminate p from (2.3)–(2.5), if we neglect $o(1)$ times the magnitude of each term, whatever that magnitude is, and if we also ignore the terms containing $h(x)$ (which can easily be justified *a posteriori*, since $a_1 > 0$), those equations may be reduced as follows: from (2.3) and (2.4), using (2.10),

$$\begin{aligned} &i\overline{\omega}(xv)' + \frac{n}{x}\overline{\omega}y - i\epsilon\beta nxw \\ &\quad = \frac{\epsilon}{R}\left\{\left[xv'' + v' - \frac{v}{x}(1+n^2) + \frac{2iny}{x^2}\right]' - \frac{in}{x}\left(y'' - \frac{y'}{x} - \frac{n^2}{x^2}y - \frac{2inv}{x}\right)\right\}; \end{aligned} \tag{2.11}$$

from (2.4) and (2.5),

$$\begin{aligned} &i\epsilon\overline{\omega}\beta xv - i\overline{\omega}w + \epsilon(\beta+2)\,y \\ &\quad = \frac{\epsilon}{R}\left\{-\left(w'' + \frac{w'}{x} - \frac{n^2w}{x^2}\right) + \epsilon\beta x\left[v'' + \frac{v'}{x} - \frac{v}{x^2}(1+n^2) + \frac{2iny}{x^3}\right]\right\}. \end{aligned} \tag{2.12}$$

At this point we assume that R_c tends neither to zero nor to infinity as $\epsilon \to 0$, so we can restrict attention to values of R such that both $R = O(1)$ and $1/R = O(1)$. Such restrictions are physically plausible, since we do not expect instability to occur at very low Reynolds numbers, but we do expect it to occur for some finite values of R, however small ϵ is. In any case, if we cannot find R_c under these restrictions, we may relax them and try again. The right-hand sides of (2.11) and (2.12) are thus of order ϵ.

Let us postulate asymptotic expansions for the dependent variables as ϵ tends to zero:

$$y(x) = y_0(x)[1+o(1)], \quad v(x) = v_0(x)[1+o(1)], \quad w(x) = w_0(x)[1+o(1)], \tag{2.13}$$

in which we assert that y_0, v_0 and w_0 are of the same order of magnitude. This assertion can be justified as follows: (i) we may assume that $v_0 \sim y_0$ (where

'$A \sim B$' means 'A has the same order of magnitude as B when $\epsilon \to 0$'), for, if $y_0 = o(v_0)$, say, then (2.10) requires $w_0 \sim v_0/\epsilon$. Then (2.12) either implies $\overline{\omega} \sim \epsilon$, in which case (2.11) gives $w_0 \equiv 0$, or reduces to

$$w_0'' + \frac{w_0'}{x} - n^2 \frac{w_0}{x^2} = 0,$$

whose only solution both non-singular at $x = 0$ and zero at $x = 1$ is $w_0 \equiv 0$; but $w_0 \equiv 0$ means $v_0 \equiv 0$, from (2.10), which contradicts the original assumption. The possibility $v_0 = o(y_0)$ is ruled out in the same way; (ii) similar arguments also rule out the possibility that w_0 is large (i.e. $w_0 \sim v_0/\epsilon \sim y_0/\epsilon$). Hence $w_0 = O(v_0)$; (iii) however, if $w_0 \sim \epsilon v_0$, then (2.12) implies that $\overline{\omega} \sim 1$, and so (2.11) reduces to

$$i(xv_0)' + ny_0/x = 0.$$

Combining this with (2.10), which reduces to

$$y_0' + inv_0 = 0, \tag{2.14}$$

we conclude that there is no non-trivial solution for y_0 which can satisfy both boundary conditions. Thus $y_0 \equiv 0$, and again the original supposition is contradicted. Hence $w_0 \sim v_0 \sim y_0$.

If we now compare the second term in (2.12) with the other terms, we see that $\overline{\omega} \sim \epsilon$, a conclusion which was also reached in the inviscid case (see Pedley 1968); so we define a quantity ω_1, of order one, such that $\overline{\omega} = \epsilon\omega_1$.

The final set of equations governing the problem is now obtained by taking the leading term in each of (2.10), (2.11) and (2.12). (2.10) again reduces to (2.14); (2.11) is unchanged, apart from replacing $\overline{\omega}$ by $\epsilon\omega_1$, and adding a suffix zero to the functions y, v, w; and in (2.12) both the first term on the left-hand side, and the second group of terms on the right-hand side, are to be neglected as being $O(\epsilon^2)$. If we eliminate v_0 from (2.11) and (2.12) by the use of (2.14), we arrive at the following pair of equations for y_0 and w_0: from (2.11),

$$y_0^{\text{iv}} + \frac{2}{x} y_0''' - \frac{1+2n^2}{x^2} y_0'' + \frac{1+2n^2}{x^3} y_0' + \frac{n^4 - 4n^2}{x^4} y_0 = R\left\{i\omega_1\left(y_0'' + \frac{1}{x} y_0' - \frac{n^2}{x^2} y_0\right) - \beta n^2 w_0\right\}; \tag{2.15}$$

and, from (2.12),

$$w_0'' + \frac{1}{x} w_0' - \frac{n^2}{x^2} w_0 = R\{i\omega_1 w_0 - (\beta+2) y_0\}. \tag{2.16}$$

The number of independent parameters appearing in the problem can be reduced to two by suitable transformations, as follows. First, define a positive quantity μ such that

$$\mu^6 = R^2 n^2 |\beta(\beta+2)|. \tag{2.17}$$

(The cases $\beta = 0$ and $\beta = -2$ will be treated separately.) Then define the quantities

$$\theta = \frac{iR\omega_1}{\mu^2}, \quad z_0 = \frac{R\beta n^2}{\mu^4} w_0, \quad x_1 = \mu x, \tag{2.18}$$

and rewrite (2.15) and (2.16) in terms of them as follows:

$$y_0^{\mathrm{iv}} + \frac{2}{x_1} y_0''' - \frac{(1+2n^2)}{x_1^2} y_0'' + \frac{(1+2n^2)}{x_1^3} y_0' + \frac{(n^4-4n^2)}{x_1^4} y_0$$

$$= \theta\left(y_0'' + \frac{1}{x_1} y_0' - \frac{n^2}{x_1^2} y_0\right) - z_0, \qquad (2.19)$$

$$z_0'' + \frac{1}{x_1} z_0' - \frac{n^2}{x_1^2} z_0 = \theta z_0 - y_0 \operatorname{sgn}[\beta(\beta+2)]. \qquad (2.20)$$

(The prime now denotes differentiation with respect to x_1.) A final substitution, which renders equation (2.19) more manageable and points the way to an analytic solution in terms of Bessel functions, is

$$Y = y_0'' + \frac{1}{x_1} y_0' + \left(1 - \frac{n^2}{x_1^2}\right) y_0, \qquad (2.21)$$

whence (2.19) becomes

$$Y'' + \frac{1}{x_1} Y' - \left(1 + \frac{n^2}{x_1^2}\right) Y = \theta(Y - y_0) - y_0 - z_0. \qquad (2.22)$$

The boundary conditions on $y_0(x_1)$ and $z_0(x_1)$ are easily obtained from (2.7) and (2.8) by using (2.14) to eliminate v_0. They are

$$y_0(\mu) = y_0'(\mu) = z_0(\mu) = 0 \qquad (2.23a)$$

and $\quad y_0(0) = z_0(0) = 0 \quad \text{for} \quad n \neq 0, \quad y_0'(0) = 0 \quad \text{for} \quad n \neq 0, 1. \qquad (2.23b)$

The problem is now seen as an eigenvalue problem for θ, with parameters n and μ. An unstable disturbance is one which leads to a positive real part of θ, since θ is proportional to $i(\sigma + n\Omega)$, and the condition for instability is $\operatorname{Im}(\sigma) < 0$. Now, the critical Reynolds number R_c which we are seeking is the lowest value of R leading to a solution with $\operatorname{Im}(\sigma) = 0$, that is $\operatorname{Re}(\theta) = 0$; but the problem is expressed in terms of μ, defined by (2.17), so we must first seek the lowest value of μ, say $\mu_1(n)$, for which $\operatorname{Re}(\theta) = 0$ for a given n, then we must vary β to obtain the lowest value of R for that μ, and then we must repeat the calculation with different, non-zero, integral values of n. If $\beta(\beta+2)$ is negative, the value of β which leads to the lowest R for a given μ and n is -1; in that case

$$R_c = \min_n \{R_1(n)\}, \quad \text{where} \quad R_1(n) = [\mu_1(n)]^3/n. \qquad (2.24)$$

If $\beta(\beta+2)$ is positive, the relevant value of β is infinite, so $R_c = 0$ (remember that this conclusion, drawn from the approximate analysis, really means $R_c = o(1)$ as $\epsilon \to 0$). However, in Pedley (1968) it was shown that $\beta(\beta+2)$ had to be negative for an inviscid unstable disturbance, and in the next section we prove the same result in the present viscous case.

As well as seeking R_c, we are interested in the growth rates of unstable disturbances for $R > R_c$, with a view to finding the most rapidly growing disturbance. If we define the growth rate as $-\operatorname{Im}(\sigma)$, we can express it in terms of θ by means of the transformations (2.2), (2.9), (2.18) and $\overline{\omega} = \epsilon\omega_1$:

$$-\operatorname{Im}(\sigma) = \frac{2\epsilon\Omega\mu^2}{R} \operatorname{Re}(\theta) = \frac{\nu}{r_0^2} \mu^2 \operatorname{Re}(\theta). \qquad (2.25)$$

Thus, for a given $R > R_c$, we wish to find the maximum of the quantity $\mu^2 \operatorname{Re}(\theta)$ as n and β are varied; and, since $\beta(\beta+2)$ will be proved negative, the maximum of μ for given n occurs when $\beta = -1$.

3. Proof that for unstable disturbances θ is real and $\beta(\beta+2)$ is negative

Before we proceed with the explicit solution of the system (2.20)–(2.22), it will be useful to prove the following results concerning the eigenvalues θ of the problem:

(i) if $\beta(\beta+2) < 0$, then $\operatorname{Im}(\theta) = 0$;

(ii) if $\beta(\beta+2) > 0$, then $\operatorname{Re}(\theta) < 0$.

We shall also prove

(iii) if $\beta(\beta+2) = 0$, then $\operatorname{Im}(i\omega_1) = 0$, $i\omega_1 < 0$.

(This cannot be expressed in terms of θ because the transformations (2.18) break down when $\mu = 0$.)

These results show that $\beta(\beta+2)$ must be negative for an unstable disturbance, and that, in this case, θ is real. Thus we can ignore complex values of θ in the search for the critical values $\mu_1(n)$ of μ; the relation (2.24) holds between R_c and the quantities $\mu_1(n)$; and the expression (2.25) for the growth rate of an unstable disturbance becomes

$$-\operatorname{Im}(\sigma) = (\nu/r_0^2)(\mu^2\theta), \tag{3.1}$$

where μ is given by (2.17) with $\beta = -1$ for maximum growth rate at the given n. Also, unstable disturbances are stationary relative to a frame of reference rotating with the pipe's angular velocity Ω, since $\operatorname{Im}(\theta) \propto \operatorname{Re}(\sigma+n\Omega) = 0$ for such disturbances.

Proof of (i) and (ii)

Multiply (2.22) by $x_1\bar{y}_0$ (where a bar over a quantity denotes its complex conjugate) and integrate with respect to x_1 from 0 to μ. After some manipulation, involving integration by parts and use of the boundary conditions (2.23), we obtain

$$B + \int_0^\mu x_1\bar{y}_0 z_0\,dx_1 + \theta\int_0^\mu x_1\left\{|y_0'|^2 + \frac{n^2}{x_1^2}|y_0|^2\right\}dx_1 = 0, \tag{3.2}$$

where

$$B = \int_0^\mu \frac{1}{x_1}\left\{|(x_1y_0')'|^2 - \frac{2n^2}{x_1}\operatorname{Re}[y_0(x_1\bar{y}_0')'] + \frac{n^4}{x_1^2}|y_0|^2\right\}dx_1. \tag{3.3}$$

The quantity B is strictly positive, since

$$\operatorname{Re}[y_0(x_1\bar{y}_0')'] \leqslant |y_0|\,|(x_1y_0')'|,$$

whence

$$B \geqslant \int_0^\mu \frac{1}{x_1}\left\{|(x_1y_0')'| - \frac{n^2}{x_1}|y_0|\right\}^2 dx_1 \geqslant 0,$$

and $B = 0$ only if y_0 has the form $A_1x^n + A_2x^{-n}$, which is prohibited by the boundary conditions.

Next we turn to equation (2.20). Multiply the complex conjugate of (2.20) by x_1z_0, and integrate from 0 to μ. A single integration by parts results in

$$C + \bar{\theta}\int_0^\mu x_1|z_0|^2dx_1 - \operatorname{sgn}[\beta(\beta+2)]\int_0^\mu x_1\bar{y}_0z_0\,dx_1 = 0, \tag{3.4}$$

where $$C = \int_0^\mu x_1 \left\{ |z_0'|^2 + \frac{n^2}{x_1^2} |z_0|^2 \right\} dx_1 > 0.$$

Finally, eliminate $$\int_0^\mu x_1 \bar{y}_0 z_0 \, dx_1$$
from (3.2) and (3.4), to obtain

$$B + \operatorname{sgn}[\beta(\beta+2)] \left\{ C + \bar{\theta} \int_0^\mu x_1 |z_0|^2 dx_1 \right\} + \theta \int_0^\mu x_1 \left\{ |y_0'|^2 + \frac{n^2}{x_1^2} |y_0|^2 \right\} dx_1 = 0. \tag{3.5}$$

(i) When $\beta(\beta+2) < 0$, take the imaginary part of (3.5):

$$\operatorname{Im}(\theta) \int_0^\mu x_1 \left\{ |y_0'|^2 + \frac{n^2}{x_1^2} |y_0|^2 + |z_0|^2 \right\} dx_1 = 0,$$

whence $\operatorname{Im}(\theta) = 0$, since the integral is positive for a non-trivial solution.

(ii) When $\beta(\beta+2) > 0$, take the real part of (3.5):

$$\operatorname{Re}(\theta) \int_0^\mu x_1 \left\{ |y_0'|^2 + \frac{n^2}{x_1^2} |y_0|^2 + |z_0|^2 \right\} dx_1 = -(B+C),$$

whence $\operatorname{Re}(\theta) < 0$, since B and C are both positive.

(iii) The case $\mu = 0$ is not covered by the above analysis, and we must revert to equations (2.15) and (2.16). From its definition (2.17), μ is zero if *either* $n = 0$, *or* $R = 0$, both of which we have excluded, *or* $\beta(\beta+2) = 0$. When $\beta + 2 = 0$, we define

$$\theta' = iR\omega_1, \quad z_0 = -2Rn^2 w_0,$$

and the equations reduce to (2.19), and (2.20) without its last term, with x, θ' in place of x_1, θ in each. Multiply the equivalent of (2.20) by $x\bar{z}_0$, integrate from 0 to 1 with respect to x, and obtain

$$\theta' \int_0^1 x |z_0|^2 dx = -\int_0^1 x \left\{ |z_0'|^2 + \frac{n^2}{x^2} |z_0|^2 \right\} dx,$$

whence either $z_0 \equiv 0$, or θ' is real and negative. If $z_0 \equiv 0$, we may multiply (2.19) by $x\bar{y}_0$, integrate from 0 to 1, and obtain (cf. (3.2))

$$\theta' \int_0^1 x \left\{ |y_0'|^2 + \frac{n^2}{x^2} |y_0|^2 \right\} dx = -B,$$

where B is defined as in (3.3), with 1 for μ and x for x_1. Thus θ' is real and negative. Similarly, when $\beta = 0$, we define

$$\theta' = iR\omega_1, \quad z_0 = w_0/2R,$$

and the equations become (2.19) without its last term, and (2.20) with $-y_0$ as its last term, and x, θ' in place of x_1, θ in each. In this case, multiplying the equivalent of (2.19) by $x\bar{y}_0$, and integrating from 0 to 1 with respect to x, gives $y_0 \equiv 0$ or θ' is real and negative. If $y_0 \equiv 0$, (2.20) then implies that θ' is real and negative. Thus, if $\beta(\beta+2) = 0$, $i\omega_1$ is real and negative.

4. Solution of the eigenvalue problem

The form of equations (2.20)–(2.22), and of the boundary conditions (2.23*b*), indicates that their complete solution might be obtainable in terms of Bessel functions of the first kind, of order n (the corresponding Bessel functions of the second kind would be prohibited by their singularity at the origin). We therefore try a solution of the form

$$Y = aJ_n(px_1), \quad y_0 = bJ_n(px_1), \quad z_0 = cJ_n(px_1).$$

The three equations are satisfied if

$$\left.\begin{aligned} -p^2c &= \theta c + b, \quad \text{from (2.20)}, \\ a &= (1-p^2)b, \quad \text{from (2.21)}, \\ -(1+p^2)a &= -b-c+\theta(a-b), \quad \text{from (2.22)}. \end{aligned}\right\} \tag{4.1}$$

Note that we have taken $\beta(\beta+2)$ to be negative, since only then can instability occur. The equations (4.1) for a, b, c are self-consistent if and only if the 3×3 determinant of the coefficients of a, b, c is zero; that is, if and only if p satisfies an equation which can be reduced to

$$p^2(p^2+\theta)^2 = 1. \tag{4.2}$$

Equation (4.2) has six roots altogether; however, three of them are merely the negatives of the other three, and $J_n(px) = (-1)^n J_n(-px)$, so without loss of generality we may choose the three roots with positive real parts. Since, for instability, θ is real and positive, one of these roots, say p_1, is real, and satisfies

$$p_1^3 + \theta p_1 - 1 = 0. \tag{4.3a}$$

The other two roots, say p_2 and $\bar{p}_2$, are complex conjugates, satisfying

$$p_2^3 + \theta p_2 + 1 = 0. \tag{4.3b}$$

We may take p_2 to have a (strictly) positive imaginary part.

The most general solution for y_0 and z_0, then, which also satisfies the boundary conditions (2.23*b*) at $x_1 = 0$, is

$$\left.\begin{aligned} y_0 &= b_1J_n(p_1x_1) + b_2J_n(p_2x_1) + b_3J_n(\bar{p}_2x_1), \\ z_0 &= -\frac{b_1J_n(p_1x_1)}{\theta+p_1^2} - \frac{b_2J_n(p_2x_1)}{\theta+p_2^2} - \frac{b_3J_n(\bar{p}_2x_1)}{\theta+\bar{p}_2^2} \\ &= -b_1p_1J_n(p_1x_1) + b_2p_2J_n(p_2x_1) + b_3\bar{p}_2J_n(\bar{p}_2x_1), \end{aligned}\right\} \tag{4.4}$$

where use has been made of the first of equations (4.1), and of both the relations (4.3). The boundary conditions (2.23*a*) at $x_1 = \mu$ (that is,

$$y_0(\mu) = y_0'(\mu) = z_0(\mu) = 0)$$

can be simultaneously satisfied by the solutions (4.4), with non-zero values of b_1, b_2, b_3, if and only if the quantities θ and μ are related by the determinantal equation

$$\begin{vmatrix} J_n(p_1\mu) & J_n(p_2\mu) & J_n(\bar{p}_2\mu) \\ p_1J_n'(p_1\mu) & p_2J_n'(p_2\mu) & \bar{p}_2J_n'(\bar{p}_2\mu) \\ -p_1J_n(p_1\mu) & p_2J_n(p_2\mu) & \bar{p}_2J_n(\bar{p}_2\mu) \end{vmatrix} = 0$$

(p_1 and p_2 being given by (4.3)). This equation is reducible to

$$F_n(\mu,\theta) \equiv \mathrm{Im}\left\{p_2 p_1 \frac{J_n'(p_1\mu)}{J_n(p_1\mu)} + (p_1+\bar{p}_2)\,p_2 \frac{J_n'(p_2\mu)}{J_n(p_2\mu)}\right\} = 0. \qquad (4.5)$$

The lowest value of μ corresponding to a neutral disturbance ($\theta = 0$, and hence from (4.3a) $p_1 = 1$), for a given n, is the first positive zero $\mu_1(n)$ of the function $F_n(\mu, 0)$. (The solution $\mu = 0$ is excluded by §3.) Also, for a given n and $R\,(>R_c)$, and hence for a given $\mu > \mu_1(n)$, the highest growth rate is given by the largest value of θ (and hence, from (4.3a), the smallest value of p_1) satisfying (4.5). Thus, for both $\theta = 0$ and $\mu > \mu_1(n)$, we seek the smallest value of $p_1\mu$ for which (4.5) is satisfied.

We may express (4.5) more concisely if we write $p_1\mu = x$ (real) and $p_2\mu = y$ (complex), and if we define the function

$$L_n(z) = zJ_n'(z)/J_n(z). \qquad (4.6)$$

Thus (4.5) may be rewritten as

$$\hat{F}_n(x) \equiv \mathrm{Im}\{yL_n(x) + (x+\bar{y})\,L_n(y)\} = 0, \qquad (4.7)$$

since y is itself a function of x, with strictly positive imaginary part, from (4.3): in the case of neutral disturbances ($\theta = 0$, $\mu = x$)

$$y = y_1(x) \equiv \tfrac{1}{2}x(1+i\sqrt{3}); \qquad (4.8a)$$

and in the case of amplified disturbances ($\theta > 0$; μ, given, $> \mu_1(n)$)

$$y = y_2(x) \equiv \tfrac{1}{2}x + i\left(\frac{\mu^3}{x} - \frac{x^2}{4}\right)^{\frac{1}{2}}. \qquad (4.8b)$$

In both cases, we seek the first zero of $\hat{F}_n(x)$.

5. Results

We first prove an analytical result, to wit that the first zero of $\hat{F}_n(x)$ lies between the first positive zero $j_{n,1}$ of $J_n(x)$ and the second positive zero $j'_{n,2}$ of $J_n'(x)$. Clearly, there is a value of x between $j_{n,1}$ and $j_{n,2}$ (the second positive zero of $J_n(x)$) for which (4.7) is satisfied, since $L_n(x) \to +\infty$ as $x \to j_{n,1}+$, and $L_n(x) \to -\infty$ as $x \to j_{n,2}-$, and $\hat{F}_n(x)$ is a continuous function for $j_{n,1} < x < j_{n,2}$ (because y is complex, and all the zeros of $J_n(z)$, i.e. all the singularities of $L_n(z)$, are real).

To show that $\hat{F}_n(x)$ has no zeros for $0 < x < j_{n,1}$, let us express $J_n(z)$ as an infinite product:

$$J_n(z) = \frac{(\frac{1}{2}z)^n}{n!} \prod_{s=1}^{\infty} \left(1 - \frac{z^2}{j_{n,s}^2}\right) \qquad (5.1)$$

(Watson 1944, p. 498), where $j_{n,s}$ is the sth positive zero of $J_n(x)$. Logarithmic differentiation of (5.1) yields

$$L_n(z) = n - 2\sum_{s=1}^{\infty} \frac{z^2}{j_{n,s}^2 - z^2}.$$

Thus
$$\hat{F}_n(x) = \operatorname{Im}(y)L_n(x) + \frac{1}{2i}[(x+\bar{y})L_n(y) - (x+y)L_n(\bar{y})]$$
$$= -2\sum_{s=1}^{\infty} A_s,$$
where
$$A_s = \operatorname{Im}(y)\frac{x^2}{j_{n,s}^2 - x^2} + \frac{1}{2i}\left[\frac{(x+\bar{y})y^2}{j_{n,s}^2 - y^2} - \frac{(x+y)\bar{y}^2}{j_{n,s}^2 - \bar{y}^2}\right]$$
$$= \operatorname{Im}(y)\left\{\frac{x^2}{j_{n,s}^2 - x^2} + \frac{j_{n,s}^2[x(y+\bar{y}) + |y|^2] + |y|^4}{|j_{n,s}^2 - y^2|^2}\right\}, \tag{5.2}$$
whence A_s is positive for all s as long as $0 < x < j_{n,s}$, since y is a complex number with positive real and imaginary parts. Thus $\hat{F}_n(x)$ is negative for all x less than $j_{n,1}$, and so its first zero lies between $j_{n,1}$ and $j_{n,2}$. Furthermore, the second term in the curly bracket of (5.2) is positive for all x, whence the second term in (4.7) is negative for all x, so that, at the first zero of $\hat{F}_n(x)$, the first term in (4.7) must be positive. In other words, this zero lies between $j_{n,1}$ and the first zero of $L_n(x)$ greater than $j_{n,1}$: that is, $j'_{n,2}$.

This result not only provides a useful starting-point for the numerical computations which follow, but also enables us to make very simple asymptotic estimates of the interesting quantities, $\mu_1(n)$ and the dimensionless growth rate $\mu^2\theta$, for large values of n.

(*a*) *Critical Reynolds number*

For integer values of n between 1 and 10, the first positive zero $\mu_1(n)$ of $F_n(\mu, 0)$ (see equation (4.5)) was computed numerically. This zero is known from the above to lie between $j_{n,1}$ and $j'_{n,2}$, so the method used was to calculate $F_n(\mu, 0)$ for a value of μ (say μ_0) slightly greater than $j_{n,1}$, then repeat with $\mu = \mu_0 + 0{\cdot}1$, $\mu_0 + 0{\cdot}2$, etc., until $F_n(\mu, 0)$ changed sign, and then to use repeated linear interpolation in order to obtain $\mu_1(n)$. The process converged rapidly, and was terminated when $\mu_1(n)$ did not change to six significant figures. The subroutine used to calculate the Bessel functions (straightforward power series in the argument) was itself accurate to six significant figures for the smaller values of n and the correspondingly modest values of $\mu_1(n)$, but for the larger values of n in the range 1–10, and the correspondingly large values of $\mu_1(n)$, its accuracy was down to three significant figures. The results are therefore given only to three significant figures, which is in any case quite sufficient for comparison with experiment.

The results are given in table 1, where $R_1(n)$ is the Reynolds number corresponding to the given values of n and μ: $R_1(n) = \mu_1^3(n)/n$, from (2.24). The quantity $\mu_e(n)$ is defined below. We can also estimate $R_1(n)$ for values of n greater than 10, because $j_{n,1} < \mu_1(n) < j'_{n,2}$, and we have the following asymptotic formulae for $j_{n,1}$ and $j'_{n,2}$:
$$\begin{matrix} j_{n,1} \\ j'_{n,2} \end{matrix} = n + \begin{matrix} 1{\cdot}856 \\ 2{\cdot}578 \end{matrix} n^{\frac{1}{3}} + \begin{matrix} 1{\cdot}033 \\ 1{\cdot}955 \end{matrix} n^{-\frac{1}{3}} + O(n^{-1}) \tag{5.3}$$
(Olver 1960, equations (1.14) and (1.18)). Thus we can write
$$\mu_1(n) = n + k_1 n^{\frac{1}{3}} + k_2 n^{-\frac{1}{3}} + O(n^{-1}), \tag{5.4}$$

where k_1 and k_2 are numbers which might depend on n, but which must lie in the ranges

$$1{\cdot}856 < k_1 < 2{\cdot}578; \quad 1{\cdot}033 < k_2 < 1{\cdot}955.$$

In fact, we can verify empirically that k_1 and k_2 are essentially independent of n. From (5.4), with $n = 9$ and $n = 10$ (neglecting the $O(n^{-1})$ term), and taking the values of $\mu_1(9)$ and $\mu_1(10)$ from table 1, we obtain

$$k_1 = 1{\cdot}95, \quad k_2 = 1{\cdot}74. \tag{5.5}$$

If we now substitute these values of k_1 and k_2 back into (5.4) with $n = 1, 2, \ldots, 8$, we obtain estimates $\mu_e(n)$ for $\mu_1(n)$, which are also given in table 1. It can be seen that these estimates are accurate to three significant figures for $n \geqslant 6$, and are even accurate to within 8 % for $n = 1$, when the $O(n^{-1})$ term is not negligible.

n	$\mu_1(n)$	$R_1(n)$	$\mu_e(n)$
1	4·36	82·9	4·69
2	5·67	91·0	5·84
3	6·91	110	7·02
4	8·12	134	8·19
5	9·31	161	9·35
6	10·5	191	10·5
7	11·6	224	11·6
8	12·8	260	12·8
9	13·9	299	13·9
10	15·0	338	15·0

Table 1

Thus (5.4) and (5.5) provide an excellent estimate for $\mu_1(n)$, and hence for $R_1(n)$, for essentially all n. The results taken together show that, for all n, $R_1(n)$ is a monotonically increasing function of n. Thus its least value, the critical Reynolds number R_c, corresponds to an azimuthal wave-number $n = 1$, and has the value

$$R_c = 82{\cdot}9.$$

(b) *Most rapidly growing disturbance*

For given $R > R_c$, we seek the largest non-dimensional growth rate $\mu^2\theta$ for all values of n such that $R > R_1(n)$, in order to see which wave-number corresponds to the most rapidly growing disturbance. The method is similar to that used in (a) above. Given R and n, and hence also given μ, we choose a value of p_1 (and hence, from (4.3), θ and p_2 are likewise uniquely chosen) such that $p_1\mu$ is slightly larger than $j_{n,1}$, and we calculate $F_n(\mu, \theta)$ from (4.5). We then iterate as before to find the smallest value of p_1 for which $F_n(\mu, \theta)$ is zero. Finally, we calculate θ from (4.3a), and hence obtain $\mu^2\theta$.

The values of R for which the computation was performed ranged from 100 to 340 in steps of 20, and the values of n used did not exceed 9. The results are given in table 2. It can be seen that, when $R = 100$, $\mu^2\theta$ is greatest for $n = 1$ (the wave-number at which instability first becomes manifest as R is increased), but, for

$R = 120$ to 180, $\mu^2\theta$ is greatest for $n = 2$; for $R = 200$ to 320, $\mu^2\theta$ is greatest for $n = 3$; and, for $R = 340$, $\mu^2\theta$ is greatest for $n = 4$. In other words, as the Reynolds number increases, the wave-number of the most rapidly growing disturbance also increases. This ties in with the results for the inviscid case (see Pedley 1968), in which the growth rate $-\text{Im}(\sigma)$ increases with n for all n, to the limit $2\epsilon\Omega$ (which, with (2.25), suggests that, in this case, $\max(\mu^2\theta) \sim R$ as $R \to \infty$). In the

	$\mu^2\theta$								
R	$n = 1$	$n = 2$	$n = 3$	$n = 4$	$n = 5$	$n = 6$	$n = 7$	$n = 8$	$n = 9$
100	4·00	3·22	—	—	—	—	—	—	—
120	8·75	10·5	4·33	—	—	—	—	—	—
140	13·5	17·8	13·2	2·98	—	—	—	—	—
160	18·4	25·1	22·1	13·0	—	—	—	—	—
180	23·3	32·5	31·1	23·1	10·2	—	—	—	—
200	28·1	39·9	40·0	33·2	21·2	4·97	—	—	—
220	33·0	47·3	49·0	43·3	32·1	16·6	—	—	—
240	38·0	54·7	58·1	53·4	43·1	28·3	9·55	—	—
260	42·9	62·1	67·1	63·6	54·1	39·9	21·8	0·979	—
280	47·8	69·6	76·1	73·8	65·1	51·6	34·0	12·8	—
300	52·8	77·1	85·2	83·9	76·2	63·3	46·2	25·5	1·39
320	57·7	84·5	94·3	94·1	87·2	75·0	58·5	38·2	14·5
340	62·7	92·0	103	104	98·3	86·8	70·8	50·9	27·6

TABLE 2

viscous case, clearly the growth rate at first increases with n (for a given R), reaches a maximum, then decreases again to zero when $R_1(n) \geqslant R$. The Reynolds numbers at which the most unstable wave-number changes from $n = 1$ to 2, 2 to 3, 3 to 4 can be estimated from figure 1, which contains the information of table 2 in the form of graphs of $\mu^2\theta$ against R for different values of n. The transitional Reynolds numbers are

$$n = 1\text{–}2, \quad R = 109,$$
$$n = 2\text{–}3, \quad R = 199,$$
$$n = 3\text{–}4, \quad R = 323.$$

These transitions, too, we can estimate for large values of n with the help of the asymptotic formula (5.4). From (4.3*a*) we have

$$\theta = \frac{1-p_1^3}{p_1} \quad \text{or} \quad \mu^2\theta = \frac{\mu^3 - p_1^3\mu^3}{p_1\mu}, \tag{5.6}$$

where $p_1\mu$ might be expected to depend on both R and n. Now $\mu^3 = Rn$, so only if $p_1\mu$ is independent of R will (5.6) represent a linear graph of $\mu^2\theta$ against R (for each n). But figure 1 shows that the graphs of $\mu^2\theta$ against R are approximately linear for all $n \leqslant 10$, and $p_1\mu \sim n$ for large n (from $j_{n,1} < p_1\mu < j'_{n,2}$ and (5.3)), so that $p_1\mu$, for each n, is the same for all $R \geqslant R_1(n)$, and hence for all $\theta \geqslant 0$. Now, when $\theta = 0$, $p_1\mu = \mu_1(n)$, so that (5.6) becomes

$$\mu^2\theta = \frac{Rn - \mu_1^3(n)}{\mu_1(n)}. \tag{5.7}$$

Equations (5.4) and (5.5) can now be used to give the estimate $\mu_c(n)$ for $\mu_1(n)$, and then (5.7) gives an expression for the growth rate as a function of R and n which is accurate for all but the smallest values of n. The Reynolds number at which the growth rate for $n = n_0 + 1$ first exceeds that for $n = n_0$ is obtained by equating the right-hand sides of (5.7) for the two values of n. For example, the transition from $n = 3$ to $n = 4$, calculated from (5.4), (5.5) and (5.7), occurs at $R = 292$, which represents an error of less than 10 %, despite the small values of

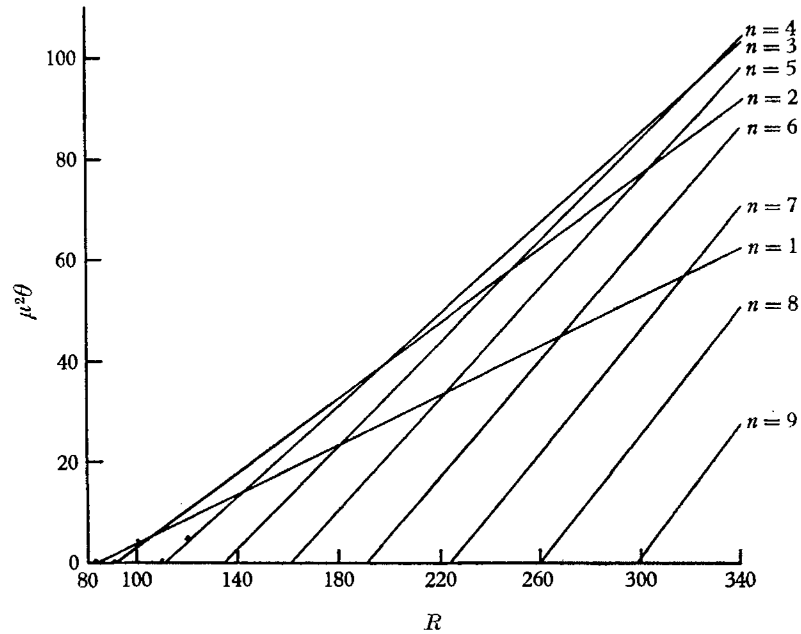

FIGURE 1. Growth rate *vs.* Reynolds number, for different values of n. Points in the lower left corner lie on the exact curves, which deviate slightly from the straight lines for small n and R.

n involved. Incidentally, if (5.7) is used with the correct values of $\mu_1(n)$, taken from table 1, the transition Reynolds number comes out as 311, an error of less than 4 %; thus for best results the correct values of $\mu_1(n)$ should be used in (5.7) when they are known and differ from the asymptotic estimates (i.e. for $n \leqslant 5$), but these estimates may be used for all $n > 5$.

6. Discussion

The principal result of this paper, that the basic flow is unstable to infinitesimal disturbances if $\epsilon \ll 1$ and $R > R_c = 82{\cdot}9$, is a surprising one for two reasons.

First, it is generally believed that non-rotating axisymmetric Poiseuille flow is stable to infinitesimal disturbances. This belief is based partly on a large body of experimental evidence (see, e.g. Leite 1959) which indicates that the Reynolds number, R_T, at which transition to turbulence occurs increases without limit as the amplitude of the disturbance tends to zero (Leite achieved stability for R

as high as 13,000), and partly on the theoretical result that the flow is stable to axisymmetric infinitesimal disturbances for all values of R (Corcos & Sellars 1959). That result is supplemented by further observations of Leite (1959) that the asymmetric part of a disturbance dies out more quickly than the symmetric part, although Fox, Lessen & Bhat (1968) have produced some evidence that non-axisymmetric disturbances (with $n = \pm 1$) are unstable for $R > 2130$.

The real cause for surprise at the present results lies in a second widespread belief, to the effect that rotation always has a stabilizing effect, whereas here its effect is clearly destabilizing. White (1964), for instance, has observed that the flow resistance in turbulent pipe flow (for various values of $R > 1000$) is much reduced as the angular velocity of the pipe is increased. White did not remark on the effect of the rotation on R_T, but the experiments of Cannon & Kays (1967) show that R_T increases as the rotation increases. This observation appears to conflict drastically with the present theory.

The resolution of the conflict must lie in the fact that the basic swirl velocity in the experiments cited was not approximately solid-body rotation. The inviscid analysis of Ludwieg (1961), who considers a flow confined to a cylindrical annulus in which the narrow gap approximation can be made, shows that a vanishingly small axial shear is sufficient to destabilize a pure, stable, swirling flow only if that flow is solid-body rotation. Ludwieg's work was extended to viscous fluids by Kiessling (1963), who found that the critical Reynolds number increases as the swirling flow deviates from solid-body rotation. Now, both White's (1964) and Cannon & Kays's (1967) experiments were conducted in rotating pipe sections at the entrance to which the flow was non-rotating. It is certainly to be expected that in such a situation, if the flow is steady, then Poiseuille flow plus solid-body rotation will be the ultimate form of the motion far downstream, but there remains the question of *how* far downstream.

If the rotation were weak, having little effect on the axial flow, the spread of axial vorticity through the fluid would follow the traditional pattern of a boundary layer whose thickness is of the order of $(\nu z/W_0)^{\frac{1}{2}}$, where z is the distance downstream. Solid-body rotation will be set up when the thickness of the layer has become comparable with the pipe radius: that is, when

$$z = O(W_0 r_0^2/\nu) = O(Rr_0). \tag{6.1}$$

The dynamics of this development is similar to that of the decay of weak rotation in a non-rotating pipe, described by Talbot (1954). However, when the rotation is strong, as in the situation envisaged here, there will be considerable interaction with the axial flow, and, only a short distance from the point of entry into the rotating section of pipe, the motion will be highly complex, dominated by rotation. A complete description of such motion, albeit steady and axisymmetric, is not available, but a lower limit to the required downstream distance can be obtained by considering the decay of small deviations from solid-body rotation.

Let the components of velocity be $[u', \Omega r + v', W_0(1 - r^2/r_0^2) + w']$, where both ϵ $(= W_0/2\Omega r_0)$ is small, and $|\mathbf{u}'| = (u'^2 + v'^2 + w'^2)^{\frac{1}{2}} \ll \Omega r_0$. Then all the inertia terms except those involving Ω (the Coriolis force terms) may be neglected in the equations of motion, and, if the pressure is eliminated between the axial and radial equations, we obtain

$$\frac{\partial}{\partial z}\{2\Omega v' + \nu(\nabla^2 u' - u'/r^2)\} = \frac{\partial}{\partial r}\{\nu\nabla^3 w'\}. \tag{6.2}$$

The azimuthal and continuity equations are, respectively,

$$2\Omega u' = \nu(\nabla^2 v' - v'/r^2) \tag{6.3}$$

and

$$\frac{1}{r}\frac{\partial}{\partial r}(ru') + \frac{\partial w'}{\partial z} = 0. \tag{6.4}$$

Let us consider the order of magnitude of the various terms in these equations, on the assumption that the length scale l of variations in the axial direction is much larger than r_0. Denote the scale of a velocity component by square brackets round it. From (6.4), we have

$$[u'] = (r_0/l)[w'] \ll [w'],$$

so that (6.5) implies

$$[v'] = \frac{\Omega r_0^3}{\nu l}[w'].$$

Hence a balance of the leading terms on either side of (6.2) yields

$$\frac{\Omega}{l}\frac{\Omega r_0^3}{\nu l}[w'] = \frac{\nu}{r_0^3}[w'],$$

or

$$l = \Omega r_0^3/\nu = O[Rr_0/\epsilon].$$

Thus the length-scale over which deviations from solid-body rotation (with or without a weak Poiseuille flow superimposed) decay is of the order Rr_0/ϵ.† When ϵ is small, this is an order of magnitude larger than the corresponding distance for weak rotation, given by (6.1) (such a result was to be expected because, in an inviscid fluid, strong rotation inhibits *all* steady axial variations, by the Taylor–Proudman theorem). If we assume that intermediate values of ϵ result in intermediate axial length scales, then, however strong the rotation, uniform solid-body rotation cannot develop until a distance of order l_0 has been travelled from the entry point, where

$$l_0 = \max[Rr_0, Rr_0/\epsilon]. \tag{6.5}$$

Now, in White's (1964) experiments, the greatest length of rotating section was 87 in., r_0 was $\frac{3}{16}$ in., and the minimum value of R at which experiments were

† This scaling arises from the balance between viscous and Coriolis forces, and is often characteristic of shear layers parallel to the axis of rotation in a rotating fluid (see, for example, Herbert 1965; Stewartson 1966).

The resulting decay problem can be solved quite simply by a separation of variables, which leads to an eigenvalue problem very similar to that solved in §4. The solution for the perturbation stream function ψ' and swirl velocity v', in the most slowly decaying mode, is

$$\psi' = C\exp(-\mu_1^3 z/2l)\,x\left\{J_1(\mu_1 x) - J_1(\mu_1)\,\mathrm{Re}\left[\frac{\omega J_1(\omega\mu_1 x)}{J_1(\omega\mu_1)}\right]\right\},$$

$$v' = \mu_1 C\exp(-\mu_1^3 z/2l)\left\{J_1(\mu_1 x) - J_1(\mu_1)\,\mathrm{Re}\left[\frac{J_1(\omega\mu_1 x)}{\omega J_1(\omega\mu_1)}\right]\right\},$$

where $x = r/r_0$, $\omega = \exp(\frac{1}{3}i\pi)$, C is a constant, and μ_1 is the first zero of $F_1(\mu, 0)$ as defined by equation (4.5). Thus $\mu_1 = \mu_1(1) = 4{\cdot}36$.

reported was, as far as can be seen from the diagrams, about 600. Thus l_0 was in all cases greater than 100 in., and solid-body rotation was presumably not attained. Also, White's minimum value of ϵ was greater than 0·1; most of the experiments were performed with values of ϵ which were not small. Similarly, in Cannon & Kays's (1967) experiments, the length of the rotating section was 60 in., r_0 was $\frac{1}{2}$ in., and the smallest value of R employed was 3000. Thus $l_0 > 1500$ in., and again solid-body rotation could not develop. Thus the present theory is inapplicable to either of these sets of experiments.

An experiment to test the theory could be performed in a pipe with a rotating section, as long as that section were long enough. One would start the section rotating with no through flow, and gradually increase the axial pressure gradient until $R = 83$ (the pipe must be rotating rapidly enough for ϵ still to be small at this value of R). If $\epsilon = 0{\cdot}1$, say, and the internal radius of the pipe is 0·1 in., then instability should be observed at a distance downstream of the order of 83 in. $\approx$ 415 diameters. (We should not expect instability to occur before this, because of Kiessling's (1963) result that the minimum critical Reynolds number corresponds to solid-body rotation.) Thus a rotating section at least 100 in. long would be required.

It might be hoped, too, that an experiment would test not only the result that $R_c \approx 83$, but also the result that the azimuthal wave-number of the most rapidly growing disturbance increases as R increases. The instability first observed would take the form of a single spiral ($n = 1$) disturbance, which is fixed relative to the rotating pipe, and which twists in the opposite sense to the streamlines in the basic flow (because $\beta = -1$, so that the axial wave-number

$$k = -\epsilon n/r_0 = -\epsilon/r_0).$$

As R is increased beyond 109, the observed disturbance should jump to double spiral ($n = 2$) form. However, the experiment outlined above would not necessarily provide these observations, for as the Reynolds number is increased we should observe the point at which instability occurs move upstream, since flows which deviate more and more from solid-body rotation would become unstable. On the other hand, the growth rates of the mode of instability associated with the solid-body rotation might be so large that this mode dominates the disturbed motion, and can still be observed. Clearly, though, the problems involved in a complete test of the theory are formidable.

It was remarked above that the unstable spiral disturbances, which would be observed in an experiment, twist in the opposite direction to the basic flow. Lambourne & Bryer (1962) observed just such a backwards spiral in vortex breakdown over a delta wing. This suggests that the present instability may be relevant to the theory of vortex breakdown (Ludwieg (1962) had the same idea). However, as Hall (1966) points out, such instability to infinitesimal disturbances can only be the first stage in the catastrophic changes, including stagnation of the axial flow, which are characteristic of true vortex breakdown. It will therefore be interesting, in the proposed experiment, to observe whether the form of the instability when its amplitude becomes large does indeed resemble the more familiar manifestations of vortex breakdown.

I am indebted to Dr M. E. McIntyre for pointing out the equivalence demonstrated in the appendix. This work was supported by the Office of Naval Research, Contract NONR 4010(02).

Appendix. Demonstration that this instability is a type of inertial instability

The classical example of inertial instability is that of pure swirling flow $[0, V(r), 0]$ in an inviscid fluid, investigated by Rayleigh (1916). The flow is unstable to axisymmetric disturbances (i.e. to disturbances whose wave-number vector $\mathbf{k}$ is parallel to the axis) if the square of the circulation anywhere decreases outwards; that is, if $(V/r^2)\, d/dr\,(Vr)$ is anywhere negative. Thus, when V is everywhere positive, the flow is unstable if the axial vorticity (the component parallel to $\mathbf{k}$) is anywhere negative.

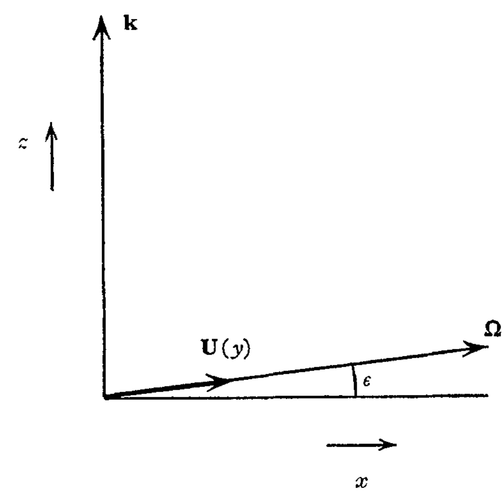

FIGURE 2. Diagram showing the relationships between the vectors $\mathbf{\Omega}$, $\mathbf{U}(y)$, and $\mathbf{k}$ in one particular flow exhibiting inertial instability. $|\mathbf{U}'| \sim \epsilon|\mathbf{\Omega}|$. The two-dimensional analogue of the instability was deduced in Pedley (1968).

The same mechanism is clearly responsible for the instability of a unidirectional shear flow $\mathbf{U}$ in a uniformly rotating fluid whose angular velocity $\mathbf{\Omega}$ is perpendicular to the direction of variation of $\mathbf{U}$. In a system of rectangular Cartesian co-ordinates (x, y, z) rotating with angular velocity $\mathbf{\Omega} = (\Omega_1, 0, \Omega_3)$, we investigate the basic flow $[U_1(y), 0, 0]$ bounded by rigid planes at $y = y_1$, $y = y_2$. Consider a disturbance whose wave-number vector $\mathbf{k} = (0, 0, k)$ is in the z-direction. The y-component of disturbance velocity may be written

$$v(y)\exp\{i(\sigma t + kz)\}. \qquad \text{(A 1)}$$

The perturbation equations of motion reduce to the following single equation for $v(y)$:

$$v'' - k^2 v + \frac{2\Omega_3 k^2(2\Omega_3 - U_1')}{\sigma^2}\, v = 0, \qquad \text{(A 2)}$$

subject to the boundary conditions $v(y_1) = v(y_2) = 0$. Sturm–Liouville theory tells us that there are negative eigenvalues for σ^2; that is there is instability if

$$\Omega_3(2\Omega_3 - U_1') < 0 \tag{A 3}$$

for any y in (y_1, y_2). Thus the flow is unstable if the z-component of absolute vorticity (the component parallel to **k**) is anywhere negative. Note that the criterion (A 3) is independent of Ω_1, which may be arbitrarily large compared with Ω_3.

This theory may be extended to show that the criterion (A 3) also results if there is a small z-component $U_3(y)$ of the basic velocity such that $|U_3| \ll |U_1|$ (assuming, on the basis of (A 3), that $|U_1'|$ and $|\Omega_3|$ are of the same order of magnitude).

Figure 2 is a diagram of the (x, z)-plane, showing the vectors $\mathbf{k}$, $\mathbf{\Omega}$, $\mathbf{U}(y)$, in the particular case where $U_3(y) = \epsilon U_1(y)$, $\Omega_3 = \epsilon\Omega_1$, and ϵ is sufficiently small that $2\Omega_3 < U_1'$ somewhere. There is a basic flow $\mathbf{U}(y)$ parallel to the rotation vector $\mathbf{\Omega}$, where $|\mathbf{U}'| \sim \epsilon|\mathbf{\Omega}|$ (since $|U_1'| \sim \Omega_3 \sim \epsilon\Omega_1$), and there are unstable disturbances with wave-number **k** almost perpendicular to $\mathbf{\Omega}$ such that the component parallel to **k** of the absolute vorticity in the basic flow is negative somewhere. Stated like that, this is clearly the two-dimensional analogue of the cylindrically symmetric problem solved in Pedley (1968).

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